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Multifractality of some random and disordered systems as a thermodynamics in fractal phase space

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Abstract. It is shown that multifractal properties of some random and disordered systems can be simulated using thermodynamics of a generalized ideal monoatomic gas in a fractal phase space.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion

1 Introduction

Constant specific heat approximation turns out to be applicable to multifractal thermodynamics [1] of some random and disordered systems [2]. It is noted in [2] that for some random systems (such as diffusion-limited aggregation and turbulent diffusion) the multifractal specific heat $c \propto d$, where d is the Euclidean dimension of the embedding space. In this paper we present some additional examples of this phenomenon (damage spreading in the Ising model and a mesoscopic system with long-range disorder at Anderson transition) and we suggest a generalization of thermodynamics of ideal monoatomic gas on *fractal* phase space as a possible model which can be used for its explanation. The ben-Avraham family of fractal gaskets [3] is used as a support of the fractal ("cell") phase space in this model.

2 Multifractal specific heat in fractal phase space

In the ordinary thermodynamics a simplest system with constant specific heat is the classical ideal monoatomic gas. For this gas, the specific heat $c = d/2$, where d is the Euclidean dimension of the embedding space [4]. Generally speaking, in the case of multifractal (virtual) thermodynamics [1] one can consider also fractal phase space. Indeed, if one considers quasi-classical motion of gas molecules, one can use the distribution of molecules in phase space instead of the distribution over quantum states. Since we shall consider only the translational motion of molecules (which are not in external field) we can use the quasi-classical approach. At this approach the statistical weight of the macroscopic state of subsystem: ΔG (the number of states corresponding to the interval ΔE of the energy in quantum theory) can be related to the volume element of phase space: $\Delta\mathbf{p}\Delta\mathbf{q}$, representing the size of region of phase space in which the subsystem will almost always be found ($\mathbf{p}$ are the momenta and $\mathbf{q}$ are the coordinates). If one supposes that a cell of volume $(2\pi h)^s$ (where s is the number of degrees of freedom of the system and h is the Planck constant) corresponds in phase space to each quantum state, then in the quasi-classical case [4]

$$\Delta G = \frac{\Delta\mathbf{p}\Delta\mathbf{q}}{(2\pi h)^s}$$

and entropy of the subsystem is

$$S = \log \frac{\Delta\mathbf{p}\Delta\mathbf{q}}{(2\pi h)^s}.$$

If the "cell"-constructed phase space is fractal with fractal dimension $2D$, then the scaling transformation of the moments and coordinates: $\mathbf{p} \to \lambda\mathbf{P}$ and $\mathbf{q} \to \lambda\mathbf{Q}$ leads to the following transformation of the volume element of the phase space $\Delta\mathbf{p}\Delta\mathbf{q} \to \lambda^{2D}\Delta\mathbf{P}\Delta\mathbf{Q}$. It is well known that the quantum states distributions can exhibit fractal properties. The quasi-classical approach allows to introduce an ideal gas with fractal properties by a natural way, while it seems to be rather difficult to do from dynamical (classical) point of view. We consider here the simplest case of monoatomic gas with energy of molecule $\varepsilon(\mathbf{p})$, which is the kinetic energy of the molecule. In this case $\varepsilon(\mathbf{p})$ is a quadratic function of the momenta $\mathbf{p}$, and the free energy of the classical (quasi-classical) gas

$$F = -NT \log \frac{A}{N} \int \mathrm{e}^{-\varepsilon(\mathbf{p})/T} \mathrm{d}\mathbf{p},$$

where N is the number of molecules in the gas, T is its temperature, and A is some constant.

In order to find the dependence on temperature of integral in this formula we substitute $\mathbf{p} \to T^{1/2}\mathbf{P}$. Since $\varepsilon(\mathbf{p})$

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is quadratic function of the momenta $\mathbf{p}$, $\varepsilon(\mathbf{p}) = T\varepsilon(\mathbf{P})$ and T cancels in the exponent of the integrand. The transformation of the d$\mathbf{p}$ gives a factor $T^{D/2}$, which can be taken outside the integral. Therefore, we obtain

$$F = -NT\log[\frac{B}{N}T^{D/2}]$$

(where B is some constant) and, consequently,

$$c = \frac{D}{2}$$

for this virtual gas.

Let us use the ben-Avraham family of fractal gaskets [3] as a support of the fractal ("cell"-constructed) phase space. This family of fractal gaskets can be constructed by the following way [3]. Take a d-dimensional hypercube of side $b = 2^n$ and subdivide it into 2^{nd} cubicles. Remove cubicles homogeneously, not randomly, until you are left with 2^{md} cubicles. Repeat the procedure with each remaining cubicles iteratively until the quasi-classical cell size. The fractal dimension of resulting object is

$$D = \frac{\log 2^{md}}{\log 2^n} = \frac{m}{n}d. \tag{1}$$

Then if we substitute the ben-Avraham's fractal dimension (1) into representation of the multifractal specific heat $c = D/2$ we obtain

$$c = \frac{m}{2n}d. \tag{2}$$

One can see that specific heat $c \propto d$ in this model. Therefore the generalized ideal gas can be used to simulate multifractal properties of the random systems for which the multifractal specific heat $c \propto d$. Moreover, choosing different m and n one can approximate any observed value of c. Therefore, this model can be used to approximate multifractal spectrum of any system with the constant multifractal specific heat $c \propto d$. The integer parameters m and n could be also used to classify these processes.

3 The examples

A. Damage spreading in the Ising model attracts constant attention in the last decade (see [5] for a review and [6] for a very recent advance). A search for multifractality in the Ising model was performed in paper [7] using the "heat bath" transition probability for Monte-Carlo simulations of the damage spreading. An equilibrium configurations was simulated in [7] and a clone made of the system. The central site was kept permanently up in the system but down in the clone: this site is permanently damaged. The authors of [7] monitor how the damage propagates with time. Whenever an update is performed it is done synchronously for site i of the system and the clone and same random number is used for the update. The use of the same random number ensures that both the system and the clone interact with the thermal reservoir in the same manner. For finite times there will be a region near the central permanently damaged site where the damage probabilities are near their equilibrium values but near the edge of the system the damaged sites have not attained their equilibrium values. Since some sites are damaged more frequently than others one can define a partition function

$$Z_q = \overline{\sum_{n=1}^{N} p_n^q} \tag{3}$$

where

$$p_n = \frac{f_n}{\sum_n^N f_n} \tag{4}$$

and f_n is the number of times site n is damaged for a fixed number of timesteps t, the bar over the symbols in (3) means average over separate realizations of the damage cluster. The sum over n is a sum over sites of the lattice with global scale L.

If there exists scaling

$$Z_q(L) \propto L^{-\tau(q)} \tag{5}$$

one can define the generalized dimensions

$$D_q = \frac{\tau(q)}{(q-1)}\,. \tag{6}$$

If D_q decreases with q we are dealing with multifractal situation. Results of numerical simulation performed in [7] indicate that the D_q is approximately constant for the Ising model both for $d = 2$ and $d = 3$, *i.e.* situation seems to be monofractal in these terms. There, however, exists also scaling

$$Z_q(t) \propto t^{-\tau(q)} \tag{7}$$

with another $\tau(q)$-function and the same numerical simulation indicates that the new $\tau(q)$ is a *nonlinear* function on q (consequently, the generalized dimensions D_q defined with this new $\tau(q)$ is non-constant). To describe the new situation quantitatively we use the constant specific heat (CSH) approximation [2]. In this approximation

$$D_q = D_\infty + c\frac{\ln q}{(q-1)} \tag{8}$$

where c is the constant multifractal specific heat. Figure 1 shows D_q/d corresponding to the t-scaling (7) and obtained in the numerical simulation of damage spreading in the Ising model [7]. Axes in this Figure are chosen for comparison with the CSH approximation (8). One can see that there is good agreement (the straight line) between the data and the CSH-approximation for both $d = 2$ and $d = 3$ cases. Moreover, it follows from this Figure that $c/d \simeq \text{const} \simeq 1/2$, *i.e.*

$$c \simeq \frac{d}{2} \tag{9}$$

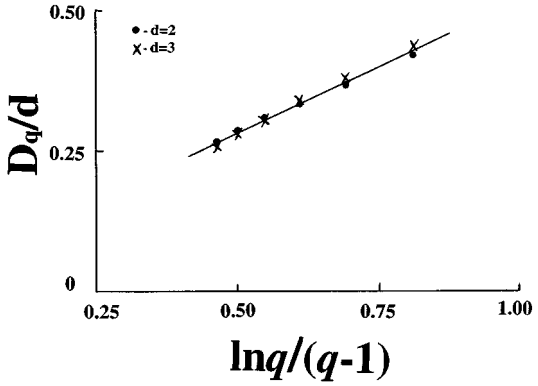

Fig. 1. Normalized generalized dimensions D_q/d against $\ln(q)/(q-1)$ for damage spreading in the Ising model. Data (symbols) taken from [7] for t-scaling. Straight line is drawn for comparison with the CSH-approximation (8).

for this model. Therefore the multifractal thermodynamics of the damage spreading in the Ising model can be interpreted as thermodynamics of a generalized ideal monoatomic gas in the phase space with the Euclidean dimension d.

B. Another interesting example of the multifractal thermodynamics of the generalized ideal gas is multifractality of wave functions of mesoscopic systems with long-range disorder at Anderson transition. In this case, however, we are dealing with the *fractal* phase space. It is shown in paper [2] that the multifractal thermodynamics of the wave functions (with short-range disorder) in a vicinity of the Anderson transition can be characterized by the generalized dimensions spectrum (8). Now the question is, whether the CSH-approximation is applicable also to the Anderson model with long-range disorder. For the long-range off-diagonal 3D disorder where the nondiagonal matrix elements $V(R)$ falling off $\propto 1/R^3$ or slower all states are delocalized [8,9] (in other dimensions, d, this result can be extended replacing $1/R^3$ by $1/R^d$). This dependence of transition matrix elements is characteristic for the dipole interaction between elastic defects in solids. Such type of interaction between soft harmonic oscillators leads to universal linear frequency dependence of the density of states above the boson peak in glasses [10]. Because of the long-range correlations these delocalized states have multifractal spatial structure causing anomalous diffusion of excitation in the system.

In a recent paper [11] a numerical simulation of such type of a system was performed. Off-diagonal disorder was introduced as $V_{ij} = (\pm 1)/|\mathbf{R}_i - \mathbf{R}_j|^d$. Here R_i are Poisson-distributed random points in d dimensional space, and the random sign ± 1 provides for the average value $\langle V_{ij}\rangle = 0$ corresponding to the interaction of randomly oriented electric or elastic dipoles. Figure 2 (adapted from [11]) shows generalized dimensions spectra obtained in this numerical simulation for the most extended eigenstates in the spaces with different dimensions: $d = 1, 2, 3$. The straight lines are drawn for comparison with the CSH-approximation (8). And again one can see from Figure 2 that $c/d \simeq$ const. In this case, however,

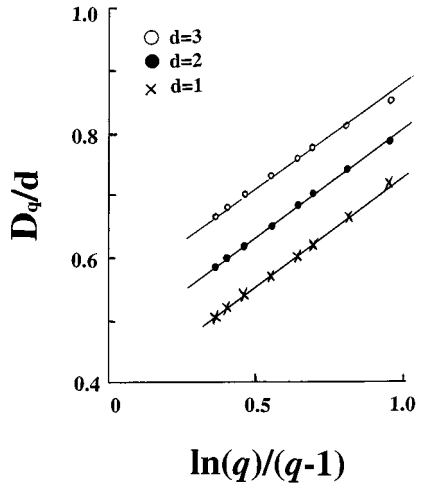

Fig. 2. Normalized generalized dimensions D_q/d against $\ln(q)/(q-1)$ for the most extended eigenstates. Data (symbols) are taken from [11]. The straight lines are drawn for comparison with the CSH-approximation (8).

$$c \simeq \frac{d}{3}. \tag{10}$$

One can approximate multifractal spectrum in this case using the generalized ideal gas model with $\{m = 2, n = 3\}$ (see (2)).

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